DENOMINATORS OF BOUNDARY SLOPES FOR (1,1)-KNOTS

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Abstract. We show that for every odd integer $n > 1$ the $(2, -3, n)$-pretzel knot is a hyperbolic (1,1)-knot whose exterior in S^3 contains an essential surface with boundary slope $2(n-1)^2/n$ and Euler characteristic $-n$.

1. Introduction

Embedded essential (i.e., incompressible and ∂-incompressible) surfaces have long been crucial to the study of 3-manifolds. The most general strategy for constructing essential surfaces in a 3-manifold M is established in [4], which shows how to associate essential surfaces to ideal points of irreducible curves in the character variety equal to the complex algebraic set of characters of $\mathrm{SL}_2\mathbb{C}$-representations of $\pi_1(M)$. This technique is especially effective in identifying the boundary slopes of M, i.e., the elements of $\pi_1(\partial M)$ represented by boundary components of essential surfaces. These in turn are helpful in understanding the Dehn fillings on M, which is important because every closed 3-manifold can be obtained by Dehn filling on the exterior of a knot or link in S^3. Results in this vein include [3] for fillings that yield manifolds with cyclic fundamental group and [1] for fillings that yield manifolds with finite fundamental group. Investigations of boundary slopes for knot exteriors in S^3 include [11] for two-bridge knots, [9] for two-bridge links, and [10] for Montesinos knots $K(p_1/q_1, \ldots, p_n/q_n)$ obtained by connecting $n \geq 3$ rational tangles of non-integral slopes $p_1/q_1, \ldots, p_n/q_n$ with each pair (p_i, q_i) relatively prime (these conditions will be assumed henceforth) in a simple cyclic pattern that yields a knot if just one q_i is even or if all q_i are odd and the number of odd p_i is odd; these are classified by the following (Classification Theorem 1.2 of [17]).

Theorem 1.1. *Montesinos knots $K(p_1/q_1, \ldots, p_n/q_n)$ with $n \geq 3$ are classified by the sum $\sum p_i/q_i$ and the vector $(p_1/q_1, \ldots, p_n/q_n) \bmod 1$ up to cyclic permutation and reversal of order.*

Two-bridge knots are the Montesinos knots with $n < 3$, and $(q_1, \ldots, q_n)$-pretzel knots are the Montesinos knots $K(1/q_1, \ldots, 1/q_n)$; we also recall the following modifications of Theorem III of [13] and Corollary 5 of [15] respectively.

Theorem 1.2. *A $(q_1, \ldots, q_n)$-pretzel knot with $n \geq 3$ is a torus knot if and only if $n = 3$ and (q_1, q_2, q_3) is a cyclic permutation of either $(-2\epsilon, 3\epsilon, 3\epsilon)$ or $(-2\epsilon, 3\epsilon, 5\epsilon)$ where $\epsilon = \pm 1$.*

Proposition 1.3. *A Montesinos knot is hyperbolic if it is not a torus knot.*

Following [14], the *tunnel number* $t(K)$ of a knot K in S^3 is the minimum number of mutually disjoint arcs $\{\tau_i\}$ properly embedded in $S^3 \smallsetminus K$ such that the exterior of $K \cup (\cup \tau_i)$ is a handlebody, and K has a *(g,b)-decomposition* if there is a genus g Heegaard splitting $\{W_1, W_2\}$ of S^3 such that K intersects each W_i

in a b-string trivial arc system; then $t(K) \leq g + b - 1$, so K has tunnel number one if it admits a (1,1)-decomposition, i.e., is a (1,1)-knot, a type of knot that has attracted much attention recently (e.g., [7] and [8] investigate closed and meridional essential surfaces in (1,1)-knot exteriors). A related result for Montesinos knots is the following taken from Theorem 2.2 and a closing remark of [14].

Theorem 1.4. *If $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$ is a Montesinos knot with $q_1 = 2$ and $q_2 \equiv q_3 \equiv 1 \bmod 2$ up to cyclic permutation of the indices, then K has tunnel number one and admits a (1,1)-decomposition.*

The denominators of boundary slopes are particularly important; e.g., [6] conjectures that there are no essential surfaces in two-bridge link exteriors with boundary in only one component and slope $1/n$ if $n \geq 6$. This in turn implies that all tunnel number one knot exteriors containing a closed essential surface of genus at least two are hyperbolic. Another result concerning the denominators of boundary slopes is the following upper bound taken from Theorem 1.1 of [12].

Theorem 1.5. *If K is a Montesinos knot with $n \geq 3$ rational tangles and is not a $(-2, 3, t)$-pretzel knot for odd $t \geq 3$, S an essential surface properly embedded in $S^3 \smallsetminus K$ with boundary slope p/q where $q > 0$ and p and q are relatively prime, and $\chi(S)$ and $\#b(S)$ the Euler characteristic and number of boundary components of S, then $q \leq -\chi(S)/\#b(S)$.*

In this paper, we use an algorithm of [10] also presented in [2], [12], and [16] to show that the upper bound in Therorem 1.5 is achieved for an infinite class of hyperbolic (1,1)-knots by proving the following.

Theorem 1.6. *For any odd integer $n > 1$, the $(2, -3, n)$-pretzel knot, which is the Montesinos knot*

$$K_n = K\left(\frac{1}{n}, \frac{2}{3}, -\frac{1}{2}\right),$$

is a hyperbolic (1,1)-knot whose exterior in S^3 contains an essential surface with boundary slope $2(n-1)^2/n$ and Euler characteristic $-n$.

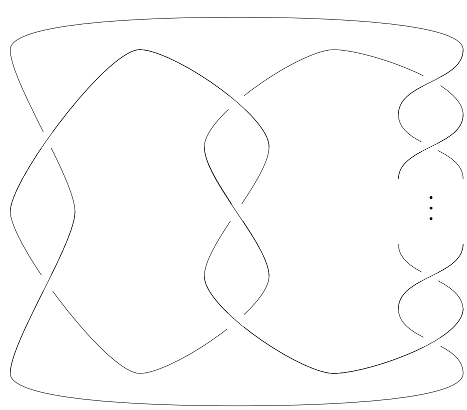


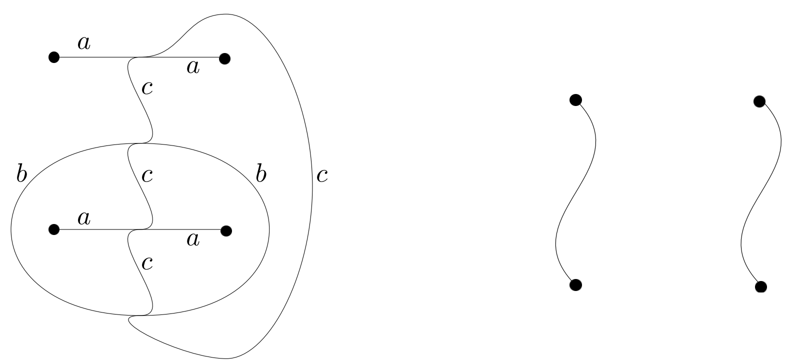
Figure 1. The $(2, -3, n)$-pretzel knot is the Montesinos knot $K(1/2, -1/3, 1/n)$ and hence K_n by Theorem 1.1.

The $(2,-3,n)$-pretzel knot in Figure 1 is the Montesinos knot $K(1/2,-1/3,1/n)$ and hence K_n by Theorem 1.1. This is not a torus knot by Theorem 1.2 and thus is hyperbolic by Proposition 1.3. Theorem 1.4 establishes that K_n has tunnel number one and admits a (1,1)-decomposition, i.e., is a (1,1)-knot. That $S^3 \setminus K_n$ contains an essential surface with boundary slope $2(n-1)^2/n$ and Euler characteristic $-n$ will be shown in Section 3 after reviewing the algorithm of [10] in Section 2.

2. Preliminaries

Let $K = K(p_1/q_1, \ldots p_n/q_n)$ be a Montesinos knot with $n \geq 3$. Our computation of boundary slopes for K follows the algorithm of [10] as also presented in [2], [12], and [16]. See those for details, but briefly [10] shows how to associate *candidate surfaces* to *admissible edgepath systems* in a graph $\mathcal{D}$ in the uv-plane whose vertices (u,v) correspond to projective curve systems $[a,b,c]$ on the 4-punctured sphere carried by the train track in Figure 2(a) via $u = b/(a+b)$ and $v = c/(a+b)$. Specifically, the vertices of $\mathcal{D}$ are:

- the ∞-*tangle* $\langle\infty\rangle$ in Figure 2(b) with uv-coordinates $(-1,0)$,
- the p/q-*circles* $\langle p/q\rangle^\circ$ whose uv-coordinates $(1,p/q)$ correspond to the projective curve system $[0,q,p]$, and
- the p/q-*tangles* $\langle p/q\rangle$ whose uv-coordinates $((q-1)/q, p/q)$ correspond to the projective curve system $[1,q-1,p]$.



(a) The train track with projective weights a, b, and c. (b) The ∞-tangle.

Figure 2. Curve systems and tangles.

If $|ps-qr| = 1$, then $[\langle p/q\rangle, \langle r/s\rangle]$ is a *non-horizontal edge* in $\mathcal{D}$ connecting $\langle r/s\rangle$ to $\langle p/q\rangle$; the remaining edges in $\mathcal{D}$ are:

- the *horizontal edges* $\left[\langle p/q\rangle, \langle p/q\rangle^\circ\right]$ connecting $\langle p/q\rangle^\circ$ to $\langle p/q\rangle$,
- the *vertical edges* $[\langle m\rangle, \langle m+1\rangle]$ connecting $\langle m+1\rangle$ to $\langle m\rangle$, and
- the *infinity edges* $[\langle\infty\rangle, \langle m\rangle]$ connecting $\langle m\rangle$ to $\langle\infty\rangle$

for any integer m; Figure 3 shows part of the graph $\mathcal{D}$.

Rational points $(p/q, r/s) \in \mathcal{D} \cap \mathbb{Q}^2$ need not be vertices of $\mathcal{D}$ and correspond to the projective curve systems $[s(q-p), sp, rq]$. If $[\langle p/q\rangle, \langle r/s\rangle]$ is a non-horizontal edge in $\mathcal{D}$, then $\frac{k}{m}\langle p/q\rangle + \frac{m-k}{m}\langle r/s\rangle$ is a rational point on this edge with coordinates

$$\left(\frac{k(q-s)+m(s-1)}{k(q-s)+ms}, \frac{k(p-r)+mr}{k(q-s)+ms}\right). \tag{1}$$

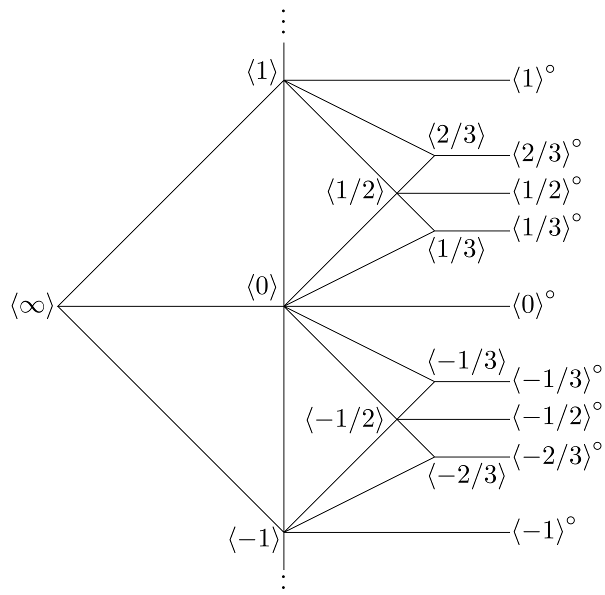


FIGURE 3. Part of the graph $\mathcal{D}$.

An *edgepath* in $\mathcal{D}$ is a piecewise linear path $[0,1] \to \mathcal{D}$ that begins and ends at rational points (not necessarily vertices) of $\mathcal{D}$. An *admissible edgepath system* $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an n-tuple of edgepaths in $\mathcal{D}$ such that:

(E1) Each starting point $\gamma_i(0)$ lies on the horizontal edge $\left[\langle p_i/q_i\rangle, \langle p_i/q_i\rangle^\circ\right]$, and γ_i is constant if $\gamma_i(0) \neq \langle p_i/q_i\rangle$.

(E2) Each γ_i is *minimal*, i.e., it never stops and retraces itself, and it never travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) The ending points $\gamma_1(1), \ldots, \gamma_n(1)$ all lie on a vertical line (i.e., have the same u-coordinates), and their v-coordinates sum to zero.

(E4) Each γ_i proceeds monotonically from right to left where traversing vertical edges is permitted, i.e., if $0 \leq t_1 < t_2 \leq 1$, then the u-coordinate of $\gamma_i(t_1)$ is at least as great as the u-coordinate of $\gamma_i(t_2)$.

The aforementioned curve systems on the 4-punctured sphere describe how the boundaries of 3-balls that decompose S^3 and each contain a tangle of K intersect properly embedded surfaces in $S^3 \smallsetminus K$, and [10] shows how to associate a finite number of *candidate surfaces* to each admissible edgepath system; their importance is the following (Proposition 1.1 in [10]).

Proposition 2.1. *Every incompressible, ∂-incompressible surface in $S^3 \smallsetminus K$ with non-empty boundary of finite slope is isotopic to one of the candidate surfaces.*

Given an admissible edgepath system $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathcal{D}$, the *final r-value* of each edgepath γ_i is the denominator of the v-coordinate at the point where the rightward extension of the final edge of γ_i intersects the vertical line $u = 1$. The sign of the final r-value is negative if this final edge travels downward from right to left. The *cycle of final r-values* of γ is the n-tuple of final r-values of the edgepaths $\gamma_1, \ldots, \gamma_n$; its importance is the following (Corollary 2.4 of [10]).

Proposition 2.2. *A candidate surface is incompressible unless the cycle of final r-values of its associated admissible edgepath system has one of the following forms: $(0, r_2, \ldots, r_n)$, $(1, \ldots, 1, r_n)$, or $(1, \ldots, 1, 2, r_n)$.*

To compute the boundary slope of a candidate surface, [10] establishes the following algorithm. The *twist number* of a candidate surface S associated to an admissible edgepath system γ is $\tau(S) = 2(e_- - e_+)$, where e_+ (e_-) is the number of edges of γ that travel upward (downward) from right to left (infinity edges are not counted). Fractional values of $e_\pm$ correspond to edges of γ that only traverse a fraction of an edge in $\mathcal{D}$, i.e., the segment from $\langle r/s\rangle$ to $\frac{k}{m}\langle p/q\rangle + \frac{m-k}{m}\langle r/s\rangle$ counts as the fraction k/m of an edge. The boundary slope of S is $\tau(S) - \tau(\Sigma)$, where Σ is a Seifert surface for K that is a candidate surface found in the following manner described on pages 460-461 of [10].

Remark 2.3. *A candidate surface associated to an admissible edgepath system $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a Seifert surface for $K(p_1/q_1, \ldots, p_n/q_n)$ if one q_i is even and each γ_i is a minimal edgepath from $\langle p_i/q_i\rangle$ to $\langle\infty\rangle$ whose* mod 2 *reduction uses only one edge of the triangle in $\mathcal{D}$ with vertices $\langle\infty\rangle$, $\langle 0\rangle$, and $\langle 1\rangle$ such that the number of odd penultimate vertices of the γ_i is even.*

To compute the Euler characteristic of a candidate surface S associated to an admissible edgepath system $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ for $K(p_1/q_1, p_2/q_2, p_3/q_3)$, we observe the following algorithm from Lemma 2.2 and the proof of Theorem 2.1 in [16]. Define the length $|\gamma_i|$ of each γ_i by counting the length of a full edge as 1 and the length of a partial edge from $\langle r/s\rangle$ to $\frac{k}{m}\langle p/q\rangle + \frac{m-k}{m}\langle r/s\rangle$ as k/m. If γ_i is not constant, let m_i be the least positive integer such that $m_i|\gamma_i| \in \mathbb{Z}$, $m = \operatorname{lcm}(m_1, m_2, m_3)$, and $\chi(\gamma_i) = m(2 - |\gamma_i|)$. Since γ is an admissible edgepath system, the ending points $\gamma_i(1)$ all have the same u-coordinate $b/(a+b)$; the Euler characteristic of S is

$$\chi(S) = \sum_{i=1}^{3} \chi(\gamma_i) - 4a - b. \tag{2}$$

3. Proof of Theorem 1.6

We now prove our result restated here for convenience.

Theorem 1.6. *For any odd integer $n > 1$, the $(2, -3, n)$-pretzel knot, which is the Montesinos knot*

$$K_n = K\left(\frac{1}{n}, \frac{2}{3}, -\frac{1}{2}\right),$$

is a hyperbolic (1,1)-knot whose exterior in S^3 contains an essential surface with boundary slope $2(n-1)^2/n$ and Euler characteristic $-n$.

Proof. Again, the $(2, -3, n)$-pretzel knot is the Montesinos knot $K(1/2, -1/3, 1/n)$ and hence K_n by Theorem 1.1; see Figure 1. This is not a torus knot by Theorem 1.2 and thus is hyperbolic by Proposition 1.3. Theorem 1.4 establishes that K_n has tunnel number one and admits a (1,1)-decomposition, i.e., is a (1,1)-knot. We now use the algorithm of [10] described in Section 2 and Formula (2) to show that $S^3 \smallsetminus K_n$ contains an essential surface with boundary slope $2(n-1)^2/n$ and Euler characteristic $-n$.

Let γ be the edgepath system in Figure 4 given by

$$\begin{aligned}
\gamma_1 &= \left[\frac{n-1}{n}\langle 0\rangle + \frac{1}{n}\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n}\right\rangle\right] \\
\gamma_2 &= \left[\frac{1}{n}\langle 0\rangle + \frac{n-1}{n}\left\langle \frac{1}{2}\right\rangle, \left\langle \frac{1}{2}\right\rangle, \left\langle \frac{2}{3}\right\rangle\right] \\
\gamma_3 &= \left[\frac{1}{n}\langle -1\rangle + \frac{n-1}{n}\left\langle -\frac{1}{2}\right\rangle, \left\langle -\frac{1}{2}\right\rangle\right].
\end{aligned}$$

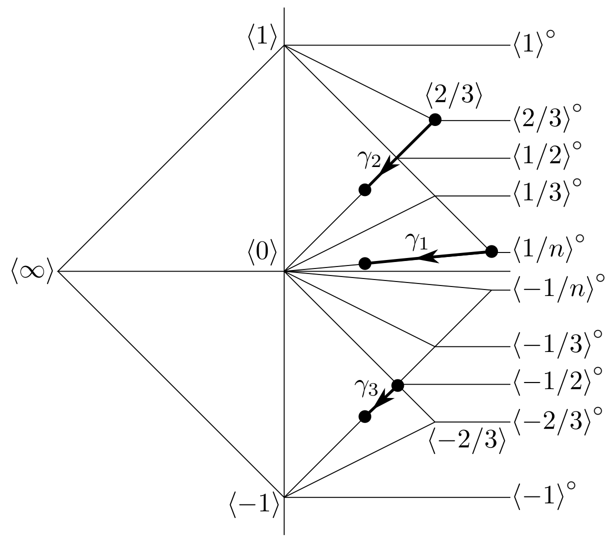


FIGURE 4. The edgepath system γ.

To obtain an associated candidate surface, we verify that γ satisfies conditions (E1-4):

(E1) $\gamma_1(0) = \langle 1/n\rangle$ lies on the horizontal edge $\left[\langle 1/n\rangle, \langle 1/n\rangle^\circ\right]$, $\gamma_2(0) = \langle 2/3\rangle$ lies on the horizontal edge $\left[\langle 2/3\rangle, \langle 2/3\rangle^\circ\right]$, and $\gamma_3(0) = \langle -1/2\rangle$ lies on the horizontal edge $\left[\langle -1/2\rangle, \langle -1/2\rangle^\circ\right]$; none of the γ_i are constant.

(E2) No γ_i stops and retraces itself or travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) Using Formula (1),

$$\begin{aligned}
\gamma_1(1) &= \frac{n-1}{n}\langle 0\rangle + \frac{1}{n}\left\langle \frac{1}{n}\right\rangle = \left(\frac{n-1}{2n-1}, \frac{1}{2n-1}\right) \\
\gamma_2(1) &= \frac{1}{n}\langle 0\rangle + \frac{n-1}{n}\left\langle \frac{1}{2}\right\rangle = \left(\frac{n-1}{2n-1}, \frac{n-1}{2n-1}\right) \\
\gamma_3(1) &= \frac{1}{n}\langle -1\rangle + \frac{n-1}{n}\left\langle -\frac{1}{2}\right\rangle = \left(\frac{n-1}{2n-1}, \frac{-n}{2n-1}\right),
\end{aligned}$$

which all lie on a vertical line (i.e., have the same u-coordinates), and their v-coordinates sum to zero.

(E4) Each γ_i proceeds monotonically from right to left.

Hence, γ is an admissible edgepath system, so a candidate surface $S_n \subset S^3 \smallsetminus K_n$ can be associated to it. To show that S_n is incompressible, we compute the cycle of final r-values of γ.

The final edge of γ_1 connects $\langle 1/n\rangle = ((n-1)/n, 1/n)$ to $\gamma_1(1)$, so its slope is $1/(n-1)$, and its rightward extension intersects the vertical line $u = 1$ at the point $(1, 1/(n-1))$. Since this final edge travels downward from right to left, the final r-value of γ_1 is $1-n$, the negative of the denominator of the v-coordinate of this point of intersection.

Similarly, the final edge of γ_2 connects $\langle 1/2\rangle = (1/2, 1/2)$ to $\gamma_2(1)$, so its slope is 1, and its rightward extension intersects the vertical line $u = 1$ at the point $(1, 1)$. Since this final edge travels downward from right to left, the final r-value of γ_2 is -1, the negative of the denominator of the v-coordinate of this point of intersection.

Lastly, the final edge of γ_3 connects $\langle -1/2\rangle = (1/2, -1/2)$ to $\gamma_3(1)$, so its slope is 1, and its rightward extension intersects the vertical line $u = 1$ at the point $(1, 0)$. Since this final edge travels downward from right to left, the final r-value of γ_1 is -1, the negative of the denominator of the v-coordinate of this point of intersection regarding 0 as $0/1$.

Thus, the cycle of final r-values of γ is $(1-n, -1, -1)$, so S_n is incompressible by Proposition 2.2. To compute its Euler characteristic, we note $|\gamma_1| = (n-1)/n$, $|\gamma_2| = (n+1)/n$, and $|\gamma_3| = 1/n$, so n is the least positive integer such that $n|\gamma_i| \in \mathbb{Z}$ for all i. Thus, $\chi(\gamma_1) = n+1$, $\chi(\gamma_2) = n-1$, and $\chi(\gamma_3) = 2n-1$. The u-coordinate of the ending points $\gamma_i(1)$ is $(n-1)/(2n-1)$, so $a = n$, $b = n-1$, and $\chi(S) = -n$ by Formula (2).

To compute the boundary slope of S_n, we note that γ_1 is $(n-1)/n$ of an edge, γ_2 is a full edge and another $1/n$ of an edge, and γ_3 is $1/n$ of an edge, all of which travel downward from right to left, so $e_+ = 0$, $e_- = (2n+1)/n$, and $\tau(S_n) = (4n+2)/n$. We now find a Seifert surface for K_n that is a candidate surface for an admissible edgepath system. Let δ be the edgepath system in Figure 5 given by

$$\begin{aligned} \delta_1 &= \left[\langle\infty\rangle, \langle 1\rangle, \left\langle\frac{1}{2}\right\rangle, \ldots, \left\langle\frac{1}{n}\right\rangle\right] \\ \delta_1 &= \left[\langle\infty\rangle, \langle 0\rangle, \left\langle\frac{1}{2}\right\rangle, \left\langle\frac{2}{3}\right\rangle\right] \\ \delta_2 &= \left[\langle\infty\rangle, \langle -1\rangle, \left\langle -\frac{1}{2}\right\rangle\right]. \end{aligned}$$

We first verify that δ satisfies conditions (E1-4):

(E1) $\delta_1(0) = \langle 1/n\rangle$ lies on the horizontal edge $\left[\langle 1/n\rangle, \langle 1/n\rangle^\circ\right]$, $\delta_2(0) = \langle 2/3\rangle$ lies on the horizontal edge $\left[\langle 2/3\rangle, \langle 2/3\rangle^\circ\right]$, and $\delta_3(0) = \langle -1/2\rangle$ lies on the horizontal edge $\left[\langle -1/2\rangle, \langle -1/2\rangle^\circ\right]$; none of the δ_i are constant.

(E2) No δ_i stops and retraces itself or travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) Each $\delta_i(1) = \langle\infty\rangle = (-1, 0)$, so they all lie on a vertical line (i.e., have the same u-coordinates), and their v-coordinates sum to zero.

(E4) Each δ_i proceeds monotonically from right to left.

Hence, δ is an admissible edgepath system with one q_i even and each δ_i a minimal edgepath from $\langle p_i/q_i\rangle$ to $\langle\infty\rangle$ with penultimate vertices $\langle 1\rangle$, $\langle 0\rangle$, and $\langle -1\rangle$ and mod 2 reductions that use only the edges $[\langle\infty\rangle, \langle 1\rangle]$, $[\langle\infty\rangle, \langle 0\rangle]$, and $[\langle\infty\rangle, \langle 1\rangle]$ respectively,

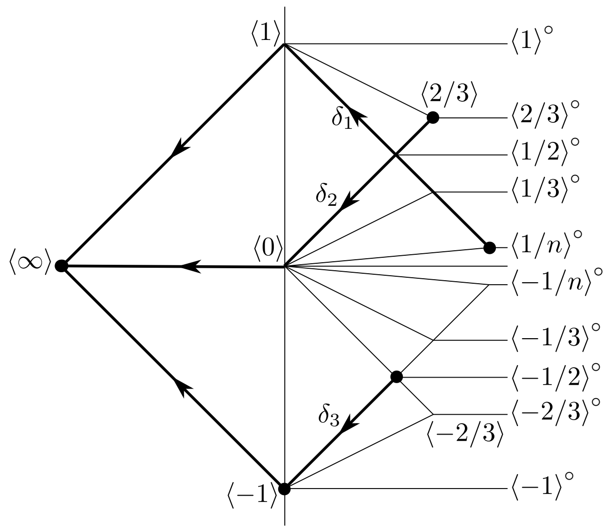


FIGURE 5. The edgepath system δ.

so an associated candidate surface Σ_n is a Seifert surface for K_n by Remark 2.3. Ignoring the infinity edges, δ_1 consists of $n-1$ edges traveling upward from right to left, δ_2 consists of two edges traveling downward from right to left, and δ_3 consists of one edge traveling downward from right to left, so $e_+ = n-1$, $e_- = 3$, and $\tau(\Sigma_n) = 8-2n$. Therefore, the boundary slope of S_n is $\tau(F_n) - \tau(\Sigma_n) = 2(n-1)^2/n$. ■

Acknowledgements. The author thanks Alan Reid for originally suggesting such a result. Nathan Dunfield's computer program to compute boundary slopes for Montesinos knots (available at `www.CompuTop.org` and described in [5]) was used in formulating and checking cases of the result. The author also thanks Eric Chesebro for assistance with this program, the algorithm of [10], and the figures.

REFERENCES

[1] S. Boyer and X. Zhang. A proof of the finite filling conjecture. *J. Differential Geom.*, 59(1):87–176, 2001.

[2] E. Chesebro and S. Tillmann. Not all boundary slopes are strongly detected by the character variety. *Comm. Anal. Geom.*, 15(4):695–723, 2007.

[3] M. Culler, C. Gordon, J. Luecke, and P. Shalen. Dehn surgery on knots. *Ann. of Math. (2)*, 125(2):237–300, 1987.

[4] M. Culler and P. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.

[5] N. Dunfield. A table of boundary slopes of Montesinos knots. *Topology*, 40(2):309–315, 2001.

[6] M. Eudave-Muñoz. Incompressible surfaces in tunnel number one knot complements. *Topology Appl.*, 98(1-3):167–189, 1999. II Iberoamerican Conference on Topology and its Applications (Morelia, 1997).

[7] M. Eudave-Muñoz. Incompressible surfaces and $(1,1)$-knots. *J. Knot Theory Ramifications*, 15(7):935–948, 2006.

[8] M. Eudave-Muñoz and E. Ramírez-Losada. Meridional surfaces and $(1,1)$-knots. *Trans. Amer. Math. Soc.*, 361(2):671–696, 2009.

[9] W. Floyd and A. Hatcher. The space of incompressible surfaces in a 2-bridge link complement. *Trans. Amer. Math. Soc.*, 305(2):575–599, 1988.

[10] A. Hatcher and U. Oertel. Boundary slopes for Montesinos knots. *Topology*, 28(4):453–480, 1989.

[11] A. Hatcher and W. Thurston. Incompressible surfaces in 2-bridge knot complements. *Invent. Math.*, 79(2):225–246, 1985.

[12] K. Ichihara and S. Mizushima. Bounds on numerical boundary slopes for Montesinos knots. *Hiroshima Math. J.*, 37(2):211–252, 2007.

[13] A. Kawauchi. Classification of pretzel knots. *Kobe J. Math.*, 2(1):11–22, 1985.

[14] K. Morimoto, M. Sakuma, and Y. Yokota. Identifying tunnel number one knots. *J. Math. Soc. Japan*, 48(4):667–688, 1996.

[15] U. Oertel. Closed incompressible surfaces in complements of star links. *Pacific J. Math.*, 111(1):209–230, 1984.

[16] Y-Q. Wu. The classification of toroidal Dehn surgeries on Montesinos knots. *Comm. Anal. Geom.*, 19(2):305–345, 2011.

[17] H. Zieschang. Classification of Montesinos knots. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 378–389. Springer, Berlin, 1984.

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